Dynamics of bright matter-wave solitons in inhomogeneous cigar-type Bose-Einstein condensate

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We discuss the possible observation of a new type of standing nonlinear atomic matter wave in the condensate: the nonlinear impurity mode. It is investigated dynamical effects of a bright soliton in Bose-Einstein-condensed (BEC) systems with local space variations of the two-body atomic scattering length. A rich dynamics is observed in the interaction between the soliton and an inhomogeneity. Processes as trapping, reflection and transmission of the bright matter soliton due to the impurity are studied considering an analytical time-dependent variational approach and also by full numerical simulation. A condition is obtained for the collapse of the bright solitary wave in the quasi-one-dimensional BEC with attractive local inhomogeneity.

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I. INTRODUCTION

Theoretical investigations of nonlinear collective excitations of matter waves, actually became a very interesting and relevant subject with the experimental observations of Bose-Einstein condensation in vapors of alkali-metal atoms [1]. One of the interesting forms of localized waves of atomic matter are the *solitons* - moving stationary nonlinear wave packets. Historically, it was first observed the *dark* solitons, represented by objects with nontrivial topological properties. They can exist in BEC systems for positive two-body scattering length ($a_s > 0$), corresponding to repulsive interactions between atoms. They can be presented as *holes* in the background of the condensates [2].

The observation of solitons in BEC systems with negative two-body scattering length ($a_s < 0$) - so called *bright* solitons, are more complicated from the experimental point of view. The difficulty is related to the instability of condensates in two-dimensions (2D) and three-dimensions (3D), when the number of atoms N exceeds the critical limit N_c, which is typically a number of the order of 1,500 atoms for Li^7. This experimental limitation can be softened in case of quasi - one-dimensional (1D) geometry; i.e., for BEC in highly anisotropic cigar-type traps. We should observe that, for a true 1D system, one does not expect the collapse of the system with increasing number of atoms [3,4]. However, it happens that a realistic 1D limit is not a true 1D system, with the density of particles still increasing due to the strong restoring forces in the perpendicular directions [5,6].

The observation of bright matter-wave solitons in BEC with attractive interactions has been recently reported in Ref. [7,8]. Theoretical models, explaining the observed phenomena, have been considered in Refs. [9,10]. It should be noted that in principle it is possible to observe the bright matter waves solitons in BEC with optical lattice. The possibility of changing the sign of the effective dispersion in such lattices makes possible to generate bright solitons for repulsive condensates [11,12]. Thus, it represents an interesting possibility to control the dynamics of bright matter-wave solitons. For the control, in this paper we suggest to use artificially induced inhomogeneities, by considering a variation in the *space* distribution of the atomic two-body scattering length. This variation can be achieved by using optical methods, as detuned laser field [13], or by means of Feshbach resonance applying an external magnetic field [14]. Mathematically, this leads to the appearance of a coordinate dependent coefficient in the nonlinear term of the Gross-Pitaevskii (GP) equation. We will analyze this problem for a cigar-type condensate using a full numerical solution of the GP formalism and also a time dependent variational approach, which was successful in the description of BEC dynamics [15,5]. In spite of the well known difficulties of the variational approach it represents a good framework for a preliminary insight into the basic physical mechanism of the model.

We consider the local space variation of the atomic scattering length as related to a *nonlinear* impurity term in the nonlinear Schrödinger equation (NLSE). Previously, interaction of solitons with linear impurities have been performed in the sine-Gordon model [16,17] and in the 1D NLSE [18–20]. It was shown that the interaction with the impurity mode leads to unexpected behaviors of the soliton. In particular, the soliton can be reflected by an *attractive* impurity. This happens due to a resonant interaction between the soliton and an oscillating impurity mode.

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For the description of the soliton dynamics, the variational approach gives two coupled equations: for the soliton width and center-of-mass position. The oscillation frequencies of the width can be transferred resonantly to the oscillation frequencies of the center of mass; and, as a result, the trapped soliton can escape from the inhomogeneity. We should also note that exists a solution representing a *standing* nonlinear atomic matter wave. The soliton, trapped by the nonlinear impurity, evolves to this solution. In this work we estimate the values of the parameters for the observation of such a new type of nonlinear standing atomic matter wave.

The structure of the present work is as follows: In Sect. 2, we formulate the model for the quasi 1D BEC with nonlinear inhomogeneity and derivate the equations for the soliton parameter using the time dependent variational approach. In Sect. 3 we present the analysis of the fixed points, frequencies of oscillations for the width and center of mass. Section 4 contains the numerical modeling of the system of ODE's and numerical simulations of the inhomogeneous GP equation, with a final discussion on the nonlinear impurity mode.

II. FORMULATION OF THE MODEL

The mean field equation for a Bose-Einstein-condensed system, trapped by an harmonic potential, is given by the following GP equation:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{m}{2}\left(\omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2\right) + \frac{4\pi\hbar^2\, a_s}{m}|\Psi(\vec{r},t)|^2\right]\Psi(\vec{r},t) \quad (1)$$

where a_s is the atomic scattering length, m is the mass of the atom, and the wave-function $\Psi \equiv \Psi(\vec{r},t)$ is normalized to the number of particles N. In the present work, we assume a cylindrical highly anisotropic trapped potential, such that i.e. $\omega_1 = \omega_2 \gg \omega_3$. In this circumstance, one can approximate the field as [5]

$$\Psi(\vec{r},t) = R(x_1,x_2)Z(x_3,t). \quad (2)$$

$R(x_1,x_2) \equiv R$ satisfies the 2D harmonic oscillator equation, that we assume is in the ground-state, normalized to one:

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + \frac{m\omega_1^2}{2}(x_1^2 + x_2^2)\right]R = \hbar\omega_1 R, \quad (3)$$

$$R(x_1,x_2) = \sqrt{\frac{m\omega_1}{\pi\hbar}}\mathrm{e}^{-\frac{m\omega_1}{2\hbar}(x_1^2+x_2^2)}. \quad (4)$$

By substituting the Eqs.(2)-(4) in Eq.(1) and averaging over the transverse coordinates,

$$\mathrm{i}\hbar\frac{\partial Z}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_3^2} + \frac{m\omega_3^2 x_3^2}{2} + \hbar\omega_1 + 2a_s\hbar\omega_1|Z|^2\right]Z. \quad (5)$$

In this case, $Z \equiv Z(x_3,t)$ is normalized to the number of particles N. Next, we redefine Z and the variables, such that

$$u(z,\tau) \equiv \sqrt{4|a_s|}Z(x_3,t)\mathrm{e}^{\mathrm{i}\omega_1 t},$$
$$\tau \equiv \frac{\omega_1 t}{2};\quad z \equiv \sqrt{\frac{m\omega_1}{\hbar}}x_3;\quad \alpha \equiv -\left(\frac{\omega_3}{\omega_1}\right)^2. \quad (6)$$

From now, we consider a simplified notation for the functions, such that $u \equiv u(z,\tau)$, $u_\tau \equiv \partial u/\partial\tau$, and $u_{zz} \equiv \partial^2 u/\partial\tau^2$. Thus, we obtain the following 1D NLSE:

$$\mathrm{i}u_\tau = -u_{zz} + \sigma|u|^2 u - \alpha z^2 u, \quad (7)$$

where σ is the signal of the two-body scattering length. The equation (7), for $\alpha = 0$ ($\omega_3 \to 0$) and $\sigma = -1$, has the solitonic solution

$$u^{(s)} = \sqrt{2}\,A\,\mathrm{sech}\,[A(z - v\tau)]\,\mathrm{e}^{\mathrm{i}\left[\frac{vz}{2} + A^2\tau - \frac{v^2\tau}{4}\right]}, \quad (8)$$

where A is a constant and v is the soliton velocity. In the following, let us consider the interesting case of local space variation of a negative atomic scattering length, such that Eq. (7) is replaced by

$$\mathrm{i}u_\tau = -u_{zz} - \alpha z^2 u - (1 + \epsilon f(z))|u|^2 u. \quad (9)$$

$\epsilon > 0$ ($\epsilon < 0$) refers to negative (positive) variation of the scattering length. The corresponding 1D Hamiltonian energy is given by

$$\langle H\rangle = \frac{1}{n_0}\int_{-\infty}^{\infty}dz\left[|u_z|^2 - \alpha z^2|u|^2 - \frac{(1+\epsilon f(z))}{2}|u|^4\right], \quad (10)$$

where n_0 is the normalization of u, related to the number of particles N and the scattering length a_s:

$$n_0 = 4N|a_s|\sqrt{\frac{m\omega_1}{\hbar}}. \quad (11)$$

We should observe that, in the realistic case one has a quasi-1D (cigar-like) trap. In Ref. [21], the authors have consider the formation and propagation of matter wave solitons, using a gas of ^{7}Li atoms, in a quasi-1D trap. The frequencies used in their trap are $\omega_\perp = \omega_1 = 2\pi\times$ 625 Hz, and $\omega_L = \omega_3 = 2\pi\times$ 3.2 Hz, with the scattering length tuned to $a_s = -3a_0$ (a_0 is the Bohr radius). The ratio of such frequencies gives $-\alpha = (\omega_3/\omega_1)^2 = 2.6\times 10^{-5}$. In this case, as shown in Ref. [6], the maximum critical number $n_{0,max} \approx 2.70$ is a constant that does not depend on a_s. The realistic maximum number of atoms N_c, will be related to a_s and the oscillator length: $4N_c|a_s|\sqrt{m\omega_1/\hbar} \approx 2.70$.

To study the dynamics of the perturbed soliton we use the following trial function [22]:

$$u = A\,\mathrm{sech}\left(\frac{z-\zeta}{a}\right)\ \mathrm{e}^{\mathrm{i}\left[\phi + w(z-\zeta) + b(z-\zeta)^2\right]}, \tag{12}$$

where A, a, ζ, ϕ, w, and b, are time-dependent variational parameters. In this case u is normalized to $n_0 = 2aA^2$. To derive the equations for the time-dependent parameters of the soliton, we first obtain the averaged Lagrangian

$$\bar{L}(\tau) = \int \mathcal{L}(z,\tau)dz, \tag{13}$$

with

$$\mathcal{L}(z,\tau) = \frac{\mathrm{i}}{2}(u_\tau u^* - u^*_\tau u) - |u_z|^2 + \alpha z^2|u|^2 + \frac{1}{2}[1+\epsilon f(z)]|u|^4. \tag{14}$$

The equations for the soliton parameters are derived from the Lagrangian $\bar{L}$, by using the corresponding Euler-Lagrange equations. So, $\bar{L}$ is given by

$$\bar{L} = -n_0\left[\phi_\tau - w\zeta_\tau + \frac{\pi^2}{12}a^2 b_\tau\right] - \frac{n_0}{3a^2} - n_0 w^2 - \frac{\pi^2 n_0\, a^2 b^2}{3} + \frac{n_0^2}{6a} + \epsilon\frac{n_0^2}{8a^2}F(a,\zeta) + \alpha\, n_0\left(\zeta^2 + \frac{\pi^2}{12}a^2\right), \tag{15}$$

where

$$F(a,\zeta) \equiv \int_{-\infty}^{\infty} dz \frac{f(z)}{\cosh^4(\frac{z-\zeta}{a})}. \tag{16}$$

We also obtain the coupled equations for a and ζ:

$$a_{\tau\tau} = \frac{16}{\pi^2 a^3} - \frac{4\,n_0}{\pi^2 a^2} - \epsilon\frac{3\,n_0}{\pi^2 a^2}\left[2\frac{F}{a} - \frac{\partial F}{\partial a}\right] + 4\alpha a,$$
$$\zeta_{\tau\tau} = 4\alpha\zeta + \epsilon\,\frac{n_0}{4a^2}\frac{\partial F}{\partial \zeta}. \tag{17}$$

When a is constant we have the well known description of the soliton center as the unit mass particle in an anharmonic potential $U(\zeta) = -2\epsilon\, A^4\mathrm{sech}^4(A\zeta)$ [23]. From this point of view, for $\epsilon < 0$, a slowly moving soliton ($v_s < v_c = 2\sqrt{|\epsilon|}$) can be trapped by the impurity. Considering the impurity with $\epsilon < 0$, the soliton can pass through the impurity or reflect, depending on its velocity. The numerical simulations of the system (17) and the GP equation (see section 4) shows that the dynamics is much more complicated. In particular we can observe the soliton reflecting from an attractive impurity even when $v_s < v_c$.

In order to have a more general formulation of the model, in the present section we have considered a non-zero external potential, parametrized by α. One could also explore the behavior of the soliton, by considering a more general time-dependent form of the external potential, as studied in Ref. [24]. However, in the present work our main motivation is the propagation of matter wave solitons, in a 1D cigar-like trap [21], such that we will assume $\alpha = 0$ in the next sections.

III. DYNAMICS OF BRIGHT SOLITONS UNDER TWO KIND OF INHOMOGENEITIES

A. Point-like non-linear impurity

Let us first consider an inhomogeneity given by a delta type ($f(z) = \delta(z)$), and look for the fixed points of the system of Eqs. (17). In this case,

$$\begin{cases} F(a,\zeta) = \mathrm{sech}^4\left(\zeta/a\right), \\ \partial F/\partial\zeta = -(1/a)\tanh\left(\zeta/a\right)\mathrm{sech}^4\left(\zeta/a\right), \\ \partial F/\partial a = (\zeta/a^2)\tanh\left(\zeta/a\right)\mathrm{sech}^4\left(\zeta/a\right). \end{cases} \tag{18}$$

The fixed point for the soliton center is given by $\zeta = 0$. This corresponds to the case of an atomic matter soliton trapped by the local variation of the two-body scattering length. In case the local variation corresponds to a positive scattering length ($\epsilon < 0$), we should observe the soliton being reflected by the inhomogeneity. Then, the stationary width a_c can be defined by

$$a_c = \frac{8 - 3\epsilon\, n_0}{2\, n_0}. \tag{19}$$

Expanding the solution near a_c we obtain the frequencies of small oscillations for the width a and for the center-of-mass ζ of the soliton, localized by the impurity. The square of such frequencies are, respectively, given by

$$\omega_a^2 = \frac{4\,n_0}{\pi^2}\left(\frac{2\,n_0}{8 - 3\,n_0\,\epsilon}\right)^3$$
$$\omega_\zeta^2 = \epsilon n_0\left(\frac{2\,n_0}{8 - 3\,n_0\,\epsilon}\right)^4. \tag{20}$$

In the variational approach for the soliton interacting with the impurity we have the interaction of the oscillating internal degree of freedom (the width) with the soliton center. As can be seen from Eqs. (20), the frequencies of the oscillations can match for a certain value of ϵ, with energy transfer between the two modes. So, as a result of the reflection from attractive inhomogeneity in BEC, a depinning of the soliton can occur.

B. Interface between two BEC media

Now, we consider another interesting case of an interface between two media, such that at $z = 0$ we have a sudden change in the two-body scattering length. The size of the inhomogeneity is given by ϵ in Eq.(9), where $f(z) = \theta(z)$. In this case, we have

$$\begin{cases} F(a,\zeta) = a\left[2/3 + \tanh(\zeta/a) - \frac{1}{3}\tanh^3(\zeta/a)\right], \\ \partial F/\partial a = F/a - (\zeta/a)\mathrm{sech}^4(\zeta/a), \\ \partial F/\partial \zeta = \mathrm{sech}^4(\zeta/a). \end{cases} \tag{21}$$

And, from (17) with $\alpha = 0$, we obtain the coupled equations:

$$a_{\tau\tau} = \frac{16}{\pi^2 a^3} - (\epsilon+2)\frac{2\,n_0}{\pi^2 a^2} - \epsilon\frac{3\,n_0}{\pi^2 a^2}\left[\tanh\left(\frac{\zeta}{a}\right)\right. \\ \left. - \frac{1}{3}\tanh^3\left(\frac{\zeta}{a}\right) + \left(\frac{\zeta}{a}\right)\mathrm{sech}^4\left(\frac{\zeta}{a}\right)\right], \tag{22}$$

$$\zeta_{\tau\tau} = \epsilon\,\frac{n_0}{4a^2}\mathrm{sech}^4\left(\frac{\zeta}{a}\right). \tag{23}$$

There is no fixed point. At the interface, the value of the width is reduced,

$$a_{int} = \frac{8}{n_0(\epsilon+2)}, \tag{24}$$

and the frequency of oscillation of the pulse width is

$$\omega_a = \frac{n_0^2(\epsilon+2)^2}{16\pi}. \tag{25}$$

For a constant value of a, from (23), we obtain

$$\zeta_\tau^2 = 2\epsilon\,\frac{n_0}{4a^2}F(a,\zeta) \le \epsilon\,\frac{2\,n_0}{3a}. \tag{26}$$

When $a = 4/n_0$ the system (22) reduces to the single equation for ζ describing the motion of the effective particle, for optical beam crossing the interface of two nonlinear Kerr media considered in [25,26]. If the velocity exceeds the critical value, the soliton pass trough the inhomogeneity. An interesting effect, predicted in Ref. [26], can occur when the soliton cross the interface, namely, the possibility of soliton splitting. The soliton is the solution in the first medium. In the second medium it can be considered as the initial wavepacket deviating from the solitonic solution, for this media. Applying the approach developed in Ref. [27], such initial condition will decay on few solitons plus radiation [26]. The number of generated solitons is equal to

$$n_{sol} = \mathcal{I}\left[\frac{1}{\sqrt{2}\pi}\int_{-\infty}^{\infty}|u_0|dz + \frac{1}{2}\right], \tag{27}$$

where u_0 is the initial solution for the second medium, and $\mathcal{I}[...]$ stands for *integer part of* [...]. Thus, if we have a jump in the scattering length given by $\Delta a_s = a_{s2} - a_{s1}$, then the number of generated solitons in the second part of the BEC is equal to $n_{sol} = \mathcal{I}\left[\sqrt{a_{s2}/a_{s1}} + 1/2\right]$, where $(1+\epsilon) = a_{s2}/a_{s1}$ is the ratio of the atomic scattering lengths. For example, for the ratio $9/4 < a_{s2}/a_{s1} < 25/4$ (or $1.25 < \epsilon < 5.25$) we obtain two solitons in the right-hand-side medium. To obtain n_{sol}, we need ϵ such that $n_{sol}(n_{sol}-1) < (\epsilon + 3/4) < n_{sol}(n_{sol}+1)$.

IV. NUMERICAL SIMULATIONS AND GENERAL DISCUSSION

Our approach for pulses deviating from the exact soliton solution is interesting from the experimental point of view, considering the difficulty in producing exact solitonic solutions. Is of particular interest the non-trivial case of nonlinear Dirac-delta impurity ($f(z) = \delta(z)$), where we made detailed comparison between the variational and full numerical solution of the GP equation. In Fig. 1, we are comparing the variation results with the numerical ones, for fixed point of the width, given by a (top frame); for the frequency of oscillations of the width, ω_a (middle frame); and for the frequency of oscillations of the center-of-mass, ω_ζ, trapped by the inhomogeneity (bottom frame). By using the variational approach, we observe that the width goes to zero and the frequencies are singular when

$$\epsilon = \epsilon_c = \frac{8}{3\,n_0}. \tag{28}$$

Here, it is interesting to observe that we have two critical numbers that are related: one of the critical number $n_{0,max} \approx 2.7$ comes from the quasi-1D limit of a 3D calculation [6]; another, is the maximum amplitude of the delta-like impurity that we have just introduced, given by (28). Considering both restrictions, we have that the smaller value of ϵ_c is about one.

The plots in Fig. 1 are valid for any value of n_0 (where the maximum is about 8/3, according to [6]), because the width and the frequencies were rescaled, such that $\epsilon' \equiv (n_0/4)\epsilon$ (implying that $\epsilon'_c = 2/3$), and

$$a' \equiv \frac{n_0}{4}a = \left(1 - \frac{3}{2}\epsilon'\right),$$

$$\omega'_a \equiv \left(\frac{4}{n_0}\right)^2\omega_a = \frac{4/\pi}{\sqrt{(1-3\epsilon'/2)^3}},$$

$$\omega'_\xi \equiv \left(\frac{4}{n_0}\right)^2\omega_\xi = \frac{2\sqrt{\epsilon'}}{(1-3\epsilon'/2)^2}.$$

As shown in Fig. 1, the variational results are supported by the full numerical calculation. The singularity occurs when the contributions coming from the inhomogeneity and nonlinearity are equal to the contribution from the quantum pressure, as seen from Eq. (17). When $\epsilon \ge \epsilon_c$, occurs the collapse of solitary wave. So, we can observe the collapse of a 1D soliton on the attractive nonlinear impurity. This possibility can be obtained following a dimensional analysis in the 1D Hamiltonian given in Eq. (10). The behavior of the field at small widths is $u \sim 1/L^{1/2}$. Taking into account that $\delta(z) \sim 1/L \sim |u|^2$, we can conclude that the contribution of the potential energy due to the impurity is $\sim |u|^6$. For positive ϵ, this term on the impurity exceeds the quantum pressure and

leads to the collapse of the soliton. In real situations, the collapse will be arrested on the final stage of the evolution, when the soliton width become of the order of the inhomogeneity scale. Then, the delta-function approximation for the impurity will break up.

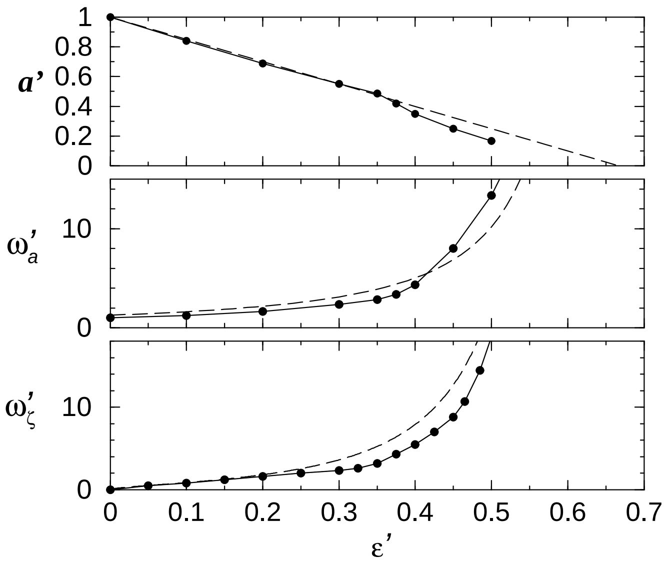

FIG. 1. The width $a' \equiv a(n_0/4)$, the frequency of the width oscillations $\omega_a' \equiv \omega_a(4/n_0)^2$, and the frequency of the soliton center oscillations $\omega_\zeta' \equiv \omega_\zeta(4/n_0)^2$, versus the strength of the nonlinear delta-like impurity $\epsilon' \equiv \epsilon(n_0/4)$. Solid line corresponds to full numerical solution of the GP equation, and dotted line to the corresponding variational approach. All the quantities are in dimensionless units, as explained in the text.

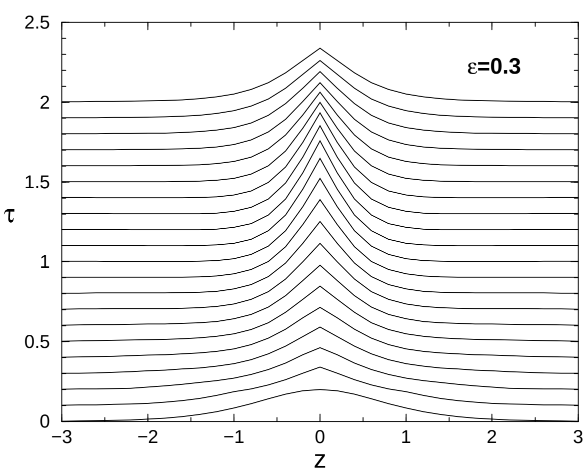

FIG. 2. Density profile evolution for a fixed value of the amplitude of the delta-like impurity, $\epsilon = 0.3$, in a projected 3D plot. At $z = 0$, we observe the oscillation of the amplitude, starting from the normal one and going to the nonlinear localized one. Each line represents a fixed value of τ. The plot is shown for a moderately small value of ϵ, in order to avoid crossing of the lines. For larger values of ϵ we can also observe the emission of radiation. τ and z are dimensionless quantities, as given in the text.

In Fig. 2, we show numerical simulations of the wave profile. We note that, after strong emission of radiation, it evolves into the so-called *nonlinear localized mode*. The nonlinear localized mode represents an *exact* solution of GP equation with nonlinear impurity (9) and it is the nonlinear *standing* atomic matter wave. The solution is given by

$$u^{(ni)} = \sqrt{2}a\text{sech}\left[a|z| + \beta\right] e^{\mathrm{i}a^2\tau}, \tag{29}$$

where

$$\beta \equiv \beta(\epsilon, a) \equiv \text{sign}(\epsilon) \ln \left(2|\epsilon|a + \sqrt{1 + 4\epsilon^2 a^2}\right)^{1/2} \tag{30}$$

This solution can be obtained by using the solution of the homogeneous equation with the requirement of the field continuity at the inhomogeneity and satisfying the jump condition in the first derivative [28]. The normalization $N^{(ni)}$, related to the umber of atoms for this solution is

$$\begin{aligned} N^{(ni)} &= 4a[1 - \epsilon\gamma], \quad \gamma \equiv \gamma(\epsilon, a) \\ \gamma &= \frac{\sqrt{1 + 4\epsilon^2 a^2} - 1}{2\epsilon^2\, a} = \frac{2\, a}{\sqrt{1 + 4\epsilon^2 a^2} + 1}. \end{aligned} \tag{31}$$

For small amplitude (or small impurity strength $|\epsilon|$) we obtain

$$N^{(ni)} \approx 4a(1 - \epsilon\, a). \tag{32}$$

At large amplitudes, we have $N^{(ni)} \to 2/\epsilon$ for $\epsilon > 0$; and $N^{(ni)} \to 8a$ for $\epsilon < 0$. Note that for $\epsilon < 0$ we have a solution with two bumps structure for the nonlinear localized mode. As shown in [28], this mode is unstable. Here, we have considered only the case $\epsilon > 0$.

In order to verify the stability of the solutions, one can study the behavior of the second time derivative of the mean-square radius, as in Refs. [4,28,29]. To obtain the second time derivative of the mean-square radius, we use the Virial approach, with $H = -\partial_{zz} + V$ and $V \equiv -(1 + \epsilon\delta(z))|u|^2$:

$$\begin{aligned} \langle z^2 \rangle_{\tau\tau} &= 4\langle [H, z\partial_z] \rangle = 8\langle (-\partial_{zz}) \rangle - 4\langle zV_z \rangle, \\ \langle zV_z \rangle &= = -\frac{1}{2}\langle V \rangle + \frac{\epsilon}{2n_0}|u_0|^4, \\ \langle z^2 \rangle_{\tau\tau} &= \frac{1}{n_0}\int dz(8|u_z|^2 - 2|u|^4) - \frac{4\epsilon}{n_0}|u_0|^4. \end{aligned} \tag{33}$$

For the system to collapse we need $\langle z^2 \rangle_{\tau\tau} < 0$; implying that

$$\epsilon > \frac{1}{2|u(0)|^4}\int (4|u_z|^2 - |u|^4)dz. \tag{34}$$

Using our solitonic ansatz, when $a \to 0$, we reach the critical limit, $\epsilon_c = 8/(3n_0)$ that was obtained before.

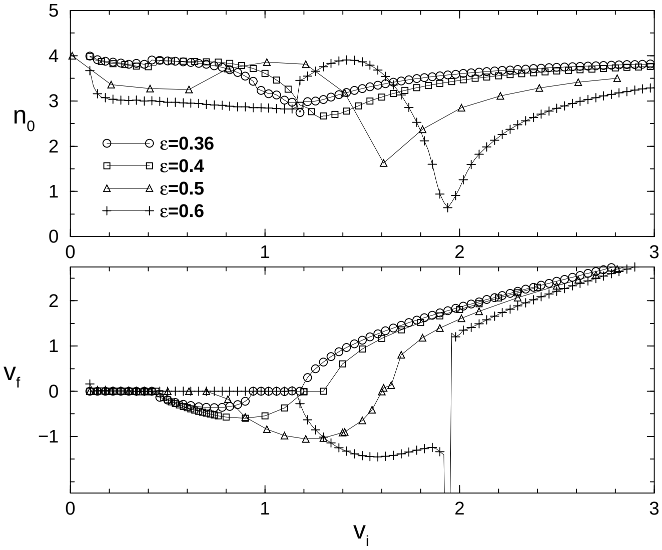

FIG. 3. Numerical simulations of the full GP equation, showing the dependence of n_0, related to the number of atoms N (top frame), and final velocity v_f (bottom frame), with respect to the initial velocity v_i. The results of both frames are shown for different values of ϵ, as indicated inside the top frame. As shown, $n_0 = 4$ was considered the initial value (for $v_i = 0$) of n_0. All the quantities are dimensionless.

We have also investigated the dynamics of the matter soliton interacting with inhomogeneity, studying different regimes of propagation for several values of ϵ. In Fig. 3, we present the results of numerical simulations for the final velocity (v_f) versus the initial velocity (v_i) of the soliton, considering different strengths ϵ for the inhomogeneity. In the present and next numerical approaches, we have considered $n_0 = 4$, for the general 1D NLSE with nonlinear impurity, as one can easily rescale the obtained data to a value of n_0 smaller than 2.7, in agreement with the quasi-1D results.

As observed in Fig. 3, exists a region for the velocities where the *attractive* nonlinear impurity reflects the soliton. In the model involving the constant width approximation, this region corresponds to the trapped soliton. The numerical results show that always exists one window corresponding to the reflection of the soliton. By increasing ϵ, this window is shifted to larger initial velocities. From the variation of the number N, with respect to the initial velocity v_i (top frame of Fig.3), one can also observe strong wave emissions by soliton, when ϵ increases and tends to the critical value (see also [28]). This picture reminds the picture of the collapse in 2D BEC with attractive interaction. We note that, by considering the interaction of sine-Gordon kink with attractive defect, many windows were found, corresponding to a resonance with local mode (see [16]).

In order to compare with the results given in the lower frame of Fig.3, we present in Fig. 4, for a fixed value of $\epsilon = 0.4$, full numerical calculation of the time evolution of the center-of-mass position, considering different values of the initial velocity. It is clearly seen the repulsive trajectories for v_i corresponding to the case that the effective particle is trapped.

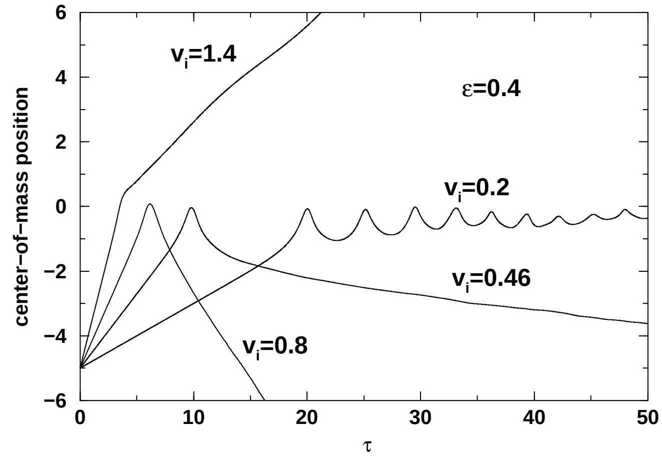

FIG. 4. For a fixed value of $\epsilon = 0.4$, it is shown the time evolution of the center-of-mass position, considering different values of the initial velocity, as given inside the frame. The results were obtained by using full numerical solution of the GP equation, considering $n_0 = 4$. All the quantities are in dimensionless units.

By considering numerical simulations of the variational equations (18) we found, qualitatively, the same behavior as the one observed in the full solution of the GP equation (see Figs. 3 and 5). In Fig. 5, we show the behavior of the final velocity as a function of the initial velocity, for different values of ϵ. In particular, the variational equations also show the existence of one window when the soliton is reflected by the impurity. When ϵ increases the window moves in the right direction, reducing the width.

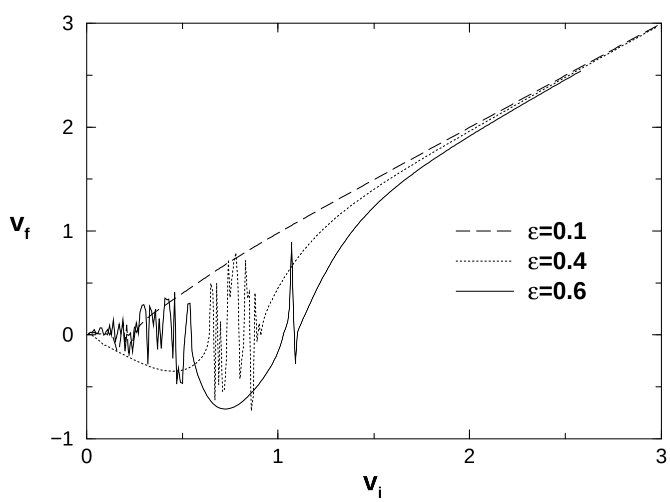

FIG. 5. Final velocity v_f versus the initial velocity v_i, for different values of ϵ, as indicated inside the frame, using $n_0 = 4$. The results were obtained from numerical simulations of the variational equations given in Eq. (17). All the quantities are in dimensionless units.

In distinction, on the GP equation, the variational equations shows a more complicated (probably chaotic) dynamics near the points where the regime of reflection starts or finishes. In this region, we also can see the more rare events with the transmission of soliton through the impurity. This observation resembles the phenomena observed at the interaction of the sine-Gordon equation kink with a local defect. The system of ODE's (two mode model) has a similar structure as our equation (18), showing chaotic behavior, leading always to a finite time for the period of the soliton trapping. This phenomenon is due to stochastic instabilities inherent for this dynamical system. The reason for this phenomenon is that the finite dimensional system, like the one given by Eq. (18), cannot take into account the soliton radiation, that interacts with the defect. The effect of the radiation leads to the appearance of the damping in Eq. (18), that changes the long-time behavior of the system. In particular, the radiative damping can lead to the long-time regular dynamics [17].

For a more detailed investigation of the interaction between matter wave soliton with the nonlinear impurity mode, excited on the inhomogeneity, we need to develop collective coordinate approach like the one considered in Ref. [18]. This will be considered in future.

V. CONCLUSION

In this work, we have investigated the dynamics of bright matter wave soliton, in BEC systems with cigar type geometry and attractive interactions. The inhomogeneities can appeared in BEC due to the existence of regions in space with different values of the two-body atomic scattering length a_s. These variations can be achieved using, for example, the Feschbach resonances. Two kind of inhomogeneities in the spatial distribution of a_s have been studied: local point-like and jump type. The first type has been modeled by a Dirac-delta function, that will result in a modulation of the nonlinear term in the GP equation, corresponding to the so called nonlinear impurity in the nonlinear Schrödinger equation. The second type corresponds to a sudden variation of the two-body scattering length, that affects the amplitude of the nonlinear term in GP equation. It correspond to the case that the BEC system is divided in two parts with different values of a_s (see discussion of recent experiment in Ref. [13]).

The present investigation of the local variation in space of the atomic scattering length shows that different regimes of the soliton interaction with the nonlinear impurity are possible. It is observed trapping, reflection and transmission regimes. The most interesting effect is the reflection of the atomic soliton by the attractive nonlinear impurity. We have also verified the occurrence of collapse of the soliton on the attractive impurity, when the strength of the impurity (or the initial number of atoms) exceeds a certain critical value. This effect in quasi-1D BEC resembles the phenomena that occurs in 2D BEC. Using the time-dependent variational approach we have described successfully both phenomena.

For the case of a sudden variation of the two-body scattering length, represented by a nonlinear jump inhomogeneity, using analogy with a nonlinear optical problem [26], we presented the condition for the multiple bright matter soliton generation.

Finally, we would like to emphasize that the problem we have studied in the present work has important application for the control of parameters of bright atomic matter solitons and for the generation of solitons in quasi 1D BEC.

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